TWO PRINCIPLES OF MAP-MAKING

By J. K. Whittemore

One of the most interesting applications of mathematics is to the problem of making a map of a part of a given surface, practically the most important surface being that of the earth considered as a sphere or as an oblate spheroid. The problem is essentially to establish a correspondence between points of the surface and points of a *plane* map, so that figures on the map shall resemble as closely as possible the corresponding figures of the surface. Since it is possible only for developable surfaces to construct a map on a constant scale,* that is such that all lengths on the surface are represented by proportional lengths on the map, it has been generally admitted by mathematicians, and to a considerable extent by geographers, since the publications on this subject of Lagrange and Gauss,† that the best maps are those which preserve the shapes of small figures, and in which, consequently, angles are unchanged. Such maps are called *conformal.*

Let us first consider the more general problem of mapping a surface S on any other surface S_1. The equations of the two surfaces may be written

$$S: \quad x = f(u, v), \qquad y = \phi(u, v), \qquad z = \psi(u, v);$$
$$S_1: \quad x_1 = f(u, v), \qquad y_1 = \phi_1(u, v), \qquad z_1 = \psi_1(u, v).$$

We may without loss of generality‡ suppose corresponding points of the two surfaces to be those obtained by giving to u and v the same values in the two sets of equations. If we denote by ds and ds_1 the "linear elements" on S and S_1, we have

$$ds^2 = dx^2 + dy^2 + dz^2 = \sum dx^2 = \sum \left(\frac{\partial x}{\partial u}\, du + \frac{\partial x}{\partial v}\, dv\right)^2$$
$$= E\, du^2 + 2F\, du\, dv + G\, dv^2,$$

* See p. 14, below.

† Lagrange, *Sur la construction des cartes géographiques.* Two memoirs presented in 1779 to the Academy of Berlin. *Oeuvres*, vol. 4, p. 637.

Gauss, *Allgemeine Auflösung der Aufgabe die Theile einer gegebenen Fläche auf einer anderen gegebenen Fläche so abzubilden, dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird;* presented in 1822 to the Royal Society of Sciences of Copenhagen, as an answer to a prize question. *Werke*, vol. 4, p. 189.

‡ Gauss, loc. cit., p. 194.

where
$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = \sum \left(\frac{\partial x}{\partial u}\right)^2,$$

$$F = \sum \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}, \quad G = \sum \left(\frac{\partial x}{\partial v}\right)^2.$$

Similarly,
$$ds_1^2 = E_1\, du^2 + 2F_1\, du\, dv + G_1\, dv^2,$$

where
$$E_1 = \sum \left(\frac{\partial x_1}{\partial u}\right)^2, \quad F_1 = \sum \frac{\partial x_1}{\partial u}\frac{\partial x_1}{\partial v}, \quad G_1 = \sum \left(\frac{\partial x_1}{\partial v}\right)^2.$$

It may be proved that it is both necessary and sufficient* for a conformal correspondence that, for all values of u and v

$$\frac{E}{E_1} = \frac{F}{F_1} = \frac{G}{G_1}.$$

If, in particular, $E = G$ and $F = 0$, so that

$$ds^2 = E(du^2 + dv^2), \tag{1}$$

and if we choose as S_1 the plane $x_1 = u$, $y_1 = v$, $z_1 = 0$, so that $ds_1^2 = du^2 + dv^2$, we shall have, by this correspondence, a conformal map of S on the plane S_1. It may be proved† that the linear element of any surface may by a suitable choice of the parameters, u and v, be thrown into the form (1) in an infinite number of ways, and further, if one such form is known, all others are obtained by writing

$$u' \pm iv' = f(u + iv),$$

where $f(u + iv)$ is any function of the complex variable $u + iv$ holomorphic in the region covered by the map.

Let us consider as examples the conformal mapping of a sphere of radius one and of an oblate spheroid obtained by rotating on its minor axis an ellipse of eccentricity e and semi-major axis one. If the center of the sphere be at the origin, and if v be the longitude measured from the xz plane and β the

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* Gauss, loc. cit., p. 195. Goursat, *Cours d'analyse mathématique*, vol. 2, p. 46.

† Gauss, loc. cit., p. 196. Darboux, *Théorie des surfaces*, vol. 1, p. 148.

complement of the latitude of a point of the sphere, we have

$$x = \sin\beta\cos v, \quad y = \sin\beta\sin v, \quad z = \cos\beta;$$

$$ds^2 = d\beta^2 + \sin^2\beta\, dv^2 = \sin^2\beta\left(\frac{d\beta^2}{\sin^2\beta} + dv^2\right).$$

Let
$$u = \int_{\pi/2}^{\beta} \frac{d\beta}{\sin\beta} = \log\tan\frac{\beta}{2}.$$

Then we have

$$\tan\frac{\beta}{2} = e^u, \qquad \cos^2\frac{\beta}{2} = \frac{1}{1+e^{2u}}, \qquad \sin^2\frac{\beta}{2} = \frac{e^{2u}}{1+e^{2u}},$$

$$\sin^2\beta = 4\sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2} = \frac{4e^{2u}}{(1+e^{2u})^2} = \left(\frac{2}{e^u+e^{-u}}\right)^2,$$

so that, finally

$$ds^2 = \left(\frac{2}{e^u+e^{-u}}\right)^2 (du^2 + dv^2).$$

The point of longitude v, latitude $90° - \beta$, is now represented by the point of coordinates (u, v) in the plane S_1 and the represention is a conformal map which may be used to represent any part of the sphere not including either pole. The map so constructed is the famous "Mercator's projection" in which circles of latitude and longitude are represented by sets of parallel straight lines, and in which a rhumb line, crossing all parallels of latitude at the same angle, is also represented by a straight line.

If we write

$$u' + iv' = e^{u+iv},$$

then is

$$u + iv = \log(u' + iv') = \log\rho + i\theta,$$

where ρ and θ are the absolute value and angle of the complex number $u' + iv'$. We have

$$u = \log\rho, \qquad v = \theta,$$

$$ds^2 = \left(\frac{2}{e^u+e^{-u}}\right)^2 (du^2 + dv^2) = \left(\frac{2}{\rho^2+1}\right)^2 (d\rho^2 + \rho^2 d\theta^2).$$

In the u', v' plane, in which ρ and θ are polar coordinates, circles of latitude of the sphere (β constant) are represented by concentric circles (ρ constant),

and circles of longitude (r constant) by straight lines radiating from the origin (θ constant). The conformal map of the sphere so constructed is that obtained by stereographic polar projection.

To construct a map of the oblate spheroid obtained by revolving about its minor axis the ellipse

$$x^2 + \frac{y^2}{1-e^2} - 1 = 0,$$

let us, following Lagrange,* express the coordinates of a point of the ellipse in terms of β, the complement of the geographical latitude, which is the angle made by the normal to the surface of revolution with the plane of the equator. One may easily see from the properties of the normal to an ellipse that $\operatorname{ctn}\beta = \dfrac{y}{x(1-e^2)}$; whence we find

$$x = \frac{\sin\beta}{\sqrt{1-e^2\cos^2\beta}}, \qquad y = \frac{(1-e^2)\cos\beta}{\sqrt{1-e^2\cos^2\beta}}.$$

If now v be the longitude of a point on the spheroid we shall have for its linear element

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + x^2\,dv^2 \\ &= \frac{\sin^2\beta}{1-e^2\cos^2\beta}\left(\frac{(1-e^2)^2}{\sin^2\beta(1-e^2\cos^2\beta)^2}\,d\beta^2 + dv^2\right). \end{aligned}$$

Let us write

$$u = \int_{\pi/2}^{\beta} \frac{(1-e^2)\,d\beta}{\sin\beta(1-e^2\cos^2\beta)} = \log\left\{\left(\frac{1+e\cos\beta}{1-e\cos\beta}\right)^{e/2}\tan\frac{\beta}{2}\right\}.$$

Then we have

$$ds^2 = \frac{\sin^2\beta}{1-e^2\cos^2\beta}\,(du^2 + dv^2).$$

The coefficient would naturally be expressed, as in the case of the sphere, in terms of u, but such an expression cannot be obtained in any simple finite

* Lagrange, loc. cit. p. 661. Lagrange and other writers use the letter z for the complement of the geographical latitude. I have used β, since in this paper z is used for one of the coordinates.

form. It is clear that these formulas reduce to those for the sphere when $e = 0$. Lagrange remarks that if we write

$$\tan\frac{\zeta}{2} = \left(\frac{1 + e\cos\beta}{1 - e\cos\beta}\right)^{e/2}\tan\frac{\beta}{2},$$

we have $u = \log\tan\zeta/2$, and ζ is an angle obtained by making a correction, which vanishes with e, in the geographical co-latitude. We have thus a map of the oblate spheroid in which circles of latitude and longitude are, as in Mercator's projection, represented by sets of parallel straight lines. The transformation considered in that case would give us on the $u'v'$ plane a map of the surface in which these curves would be represented by concentric circles and lines radiating from the center.

Having seen that any surface may be mapped conformally on a plane in various ways, we now consider the *scale* of the map, on which is to depend the choice of the best conformal map. The scale is the ratio of corresponding lengths on the map and the surface at a point. Denoting the scale of the map by m_1, we have $m_1 = ds_1/ds$. Suppose a second conformal map constructed on the $u'v'$ plane, S', where

$$u' \pm iv' = f(u + iv).$$

On this map the scale m is evidently ds'/ds. We have

$$ds^2 = E(du^2 + dv^2), \qquad ds_1^2 = du^2 + du^2,$$

$$ds'^2 = du'^2 + dv'^2 = E'(du^2 + dv^2).$$

Since

$$du' \pm i\,dv' = f'(u + iv)(du + i\,dv),$$

$$du' \mp i\,dv' = \bar{f}'(u - iv)(du - i\,dv),$$

where $\bar{f}$ is the function conjugate to f, and f' and $\bar{f}'$ denote the derivatives of f and $\bar{f}$, we have

$$E' = f'(u + iv)\bar{f}'(u - iv), \qquad m = \sqrt{\frac{E'}{E}}.$$

We have evidently

$$\log m = U - \tfrac{1}{2}\log E,$$

where

$$U = \tfrac{1}{2}\log E' = \tfrac{1}{2}\log f'(u + iv) + \tfrac{1}{2}\log\bar{f}'(u - iv)$$

and is obviously a solution of

$$\Delta U \equiv \frac{\partial^2 U}{\partial u^2} + \frac{\partial^2 U}{\partial v^2} = 0,$$

being simply the real part of the monogenic function $\log f'(u + iv)$. Moreover, a function f may be found such that U is any harmonic function, that is, a solution of $\Delta U = 0$, continuous with its partial derivatives in the region of the first map.

We may now see clearly that not every surface can be mapped on a constant scale, for, in the first place such a map must be comformal,* so we shall have

$$\Delta \log m = -\tfrac{1}{2} \Delta \log E,$$

and if m may be constant we must have $\Delta \log E = 0$, and this is not the case in either example considered. It is interesting and, for the purpose of our discussion, important to find another expression for $\Delta \log m$. Gauss obtained† an expression for the total curvature of any surface in terms of the coefficients E, F, G, and their derivatives which becomes, when $ds^2 = E(du^2 + dv^2)$,

$$K = -\frac{\Delta E}{2E^2} + \frac{1}{2E^3}\left[\left(\frac{\partial E}{\partial u}\right)^2 + \left(\frac{\partial E}{\partial v}\right)^2\right].$$

But we find

$$\Delta \log E = \frac{E\Delta E - \left[\left(\frac{\partial E}{\partial u}\right)^2 + \left(\frac{\partial E}{\partial v}\right)^2\right]}{E^2}.$$

Hence

$$\Delta \log E = -2EK \quad \text{and} \quad \Delta \log m = EK.$$

Since E is always positive, a map of a surface can have a constant scale only when $K = 0$, that is, only when the surface is developable.

The construction of a second conformal map on a plane S', has been shown to depend on the choice of a harmonic function of u and v on the map in the plane S_1. The problem which we shall now consider is the most interesting question in the theory of map-making, the choice of this harmonic

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* Gauss, loc cit., p. 194.

† *Disquisitiones generales circa superficies curvas*, presented in 1827 to the Royal Academy of Göttingen. *Werke*, vol. 4, p. 232.

function U, so that the map on the plane S' may be the best possible map. We must be guided in this choice by a consideration of the variation of the scale of the map. Various criteria for the best conformal map of a region of a surface have been proposed. Of these I shall consider fully two* which, very curiously, starting from totally different points of view reach, in most cases, the same conditions for the determination of U. The first of these principles was enunciated in 1856 by Tchébychef, in a paper presented to the Imperial Academy of Sciences of St. Petersburg,† as follows: The minimum deviation of an integral of $\Delta U = 0$ from the function

$$\log \frac{2}{e^u + e^{-u}}.$$

in a region bounded by a closed curve cannot occur unless

$$U - \log \frac{2}{e^u + e^{-u}}$$

is constant on the contour. It is not entirely obvious what Tchébychef meant by the "deviation," and his statement applies only to maps of a sphere. So far as I can learn he never published any proof of this remarkable theorem, though he applied the principle to the construction of a map of Russia. I believe that the theorem was first proved in 1894 by D. A. Gravé, a professor at St. Petersburg.‡ This proof I shall here present in a form so generalized that the principle may be applied to the construction of a map of a region of any surface in which the total curvature is throughout of one sign, thus justifying the application of the principle made in the following paper by Dr. G. W. Hill to a map of a region of an oblate spheroid.

Another criterion was proposed and discussed by F. Eisenlohr § in 1870. He regards as a measure of the accuracy of a map the distortion of the geodesic lines of the surface. If we find at each point of the map the maximum value of the curvature of the representations of all geodesic lines through

* Two other conditions for determining U were proposed by H. Weber in a paper published in *Crelle's Journal für die Mathematik*, vol. 67, p. 229, and cited by Eisenlohr. These conditions while apparently as reasonable as those of Tchébychef and Eisenlohr lead to no simple determination of U and are consequently of comparatively little interest.

† *Bulletin*, vol. 14, p. 257; *Oeuvres*, vol. 1, p. 233.

‡ *Association française pour l'avancement des sciences*, 23^e session, Caen, 1894, p. 196.

§ *Crelle's Journal für die Mathematik*, vol. 72, p. 143. This paper and Weber's are cited by Gravé, loc. cit.

this point we may determine U so that the mean value on the map of the square of this maximum curvature shall be a minimum, and find, at least in the case when the *first* map in the plane S_1 is simply connected, that this condition requires that the scale be constant on the contour. It is clear that Eisenlohr's and Tchébychef's conditions are the same when the part of the surface mapped is simply connected and has a total curvature everywhere of the same sign. I shall here give a proof of Eisenlohr's principle, supplying some details omitted by him, and emphasizing the fact, which he does not consider, that the proof applies only when the map in the plane S_1 is a simply connected region.

We now take up the proof of Tchébychef's principle. Let the region of the map in the plane S_1 be denoted by T, and the contour of this region be C. The contour C may consist of one or more curves and is considered as belonging to the region T, which is therefore "bounded." Let U be a single valued harmonic function in T, that is to say continuous with all its partial derivatives and satisfying the equation $\Delta U = 0$. We shall need the following theorems* concerning U:

a) U can have no maximum or minimum value at an interior point of T. From (a) it follows that

b) If U is negative at all points of C it is negative at all points of T. From this fact it may be proved that

c) If U vanishes at some points of C and is negative at all others, it is negative at all interior points of T.†

d) If U is constant on C, it has the same constant value at all points of T, and consequently

e) If U and U' are two harmonic functions in T equal at all points of C, then since $U - U'$ is harmonic in T and also on C, U and U' are equal at all points of T.

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* See *e. g.* a discussion of harmonic functions in Osgood's *Functionentheorie*, vol. 1, p. 543.

† Theorem (c), on which the following proof depends, is, I think, not generally stated in treatises on harmonic functions. Its truth may be easily seen as follows: under the conditions of (c) U cannot be positive at an interior point of T; if at an interior point a, U vanishes, we should have by the mean value theorem (Osgood, loc. cit., p. 544) $\int_0^{2\pi\rho} U ds = 0$, where the integral is taken around the circumference of a circle of radius ρ, wholly within T, and having its center at a. Since U is not positive at any point of the circle, it is zero at all points of the circumference, and consequently, since it is continuous, at all points of C, contrary to the hypothesis.

Gravé states without proof the following theorem: If $\phi(u)$ is, with its first and second derivatives, $\phi'(u)$ and $\phi''(u)$, continuous and single valued in T, and if $\phi''(u)$ is negative at all points of T, and if also, on C, $U - \phi(u) = 0$, where U is harmonic in T, then $U - \phi(u)$ is negative at all interior points of T. Let us prove the more general theorem:

If $\phi(u, v)$ is with its first and second partial derivatives continuous and single valued in T, and if $\Delta\phi$ is negative at all points of T, and if also, on C, $U - \phi(u, v) = 0$, where U is harmonic in T, then $U - \phi(u, v)$ is negative at all interior points of T.

Let $V = U - \phi(u, v)$. Then, at all points of T,

$$\Delta V = -\Delta\phi > 0.$$

Since V is zero on C, if it is positive at any interior point of T, there is within T a maximum value of V where necessarily

$$\frac{\partial^2 V}{\partial u^2} < 0, \qquad \frac{\partial^2 V}{\partial v^2} < 0.$$

These conditions are inconsistent with the fact that ΔV is positive at all points of T. If at any interior point V vanishes, since no displacement to a neighboring interior point can cause V to increase, we must have at this point

$$\frac{\partial V}{\partial u} = \frac{\partial V}{\partial v} = 0,$$

and then for the same reason

$$\frac{\partial^2 V}{\partial u^2} \leqq 0 \qquad \frac{\partial^2 V}{\partial v^2} \leqq 0,$$

conditions again impossible.

To apply these theorems to the proof of Tchébychef's principle, let

$$V = \log m = U - \tfrac{1}{2} \log E.$$

Then is

$$\phi(u, v) = \tfrac{1}{2} \log E, \qquad \Delta\phi = -EK.$$

Hence, if the region of the surface to be mapped is without singularities and of positive curvature throughout, the conditions of this theorem are satisfied. This is indeed the case with any region of the spheroid.

Suppose now that U has been found so that, on C, V is constant. To suppose this constant zero will affect only the size of the map, and be no essential restriction. Then suppose, on C,

$$V = \log m = U - \tfrac{1}{2} \log E = 0.$$

By the theorem just proved V is negative at all interior points of T. Let D represent the "maximum maximorum" of the absolute value of V at all points of T. Then D is the *deviation* of the map in the $u'v'$ plane, S'. The deviation in Tchébychef's principle is the difference between the greatest and least values of the logarithm of the scale, or the logarithm of the ratio of the greatest value of the scale to its least value. We may now prove that the deviation of V is less than that of V_1, where

$$V_1 = U_1 - \tfrac{1}{2} \log E,$$

and U_1 is any harmonic function in T not differing by a constant from U.

Let the maximum value of V_1 on C be a. Then $V_1 - a$ has the same deviation as V_1, and this difference is at some point of C zero, at some points certainly negative, since $U_1 - U$ is not constant, but at no point of C positive. We may write

$$V_1 - a = V + U_1 - a - U,$$

and since V vanishes at all points of C, the harmonic function $U_1 - a - U$ is zero and negative but not positive on C. Then $U_1 - a - U$ must be negative at all interior points of T, and since the same is true of V, the sum, $V_1 - a$, which vanishes at some point of C, has within T a negative value numerically greater than V. Then $V_1 - a$, and consequently V_1, have a greater deviation than V.

We may note that a map of a region of positive curvature constructed on this principle has its greatest scale on the boundary. A similar discussion shows that a map of a region of negative total curvature at each point, constructed on this principle, has a minimum deviation, and has its least scale on the boundary.

We now take up the proof of Eisenlohr's theorem. We make our point of departure two theorems concerning geodesic curvature:

1. The geodesic curvature of a geodesic line of a surface is at every point zero.

2. If the surface is plane the geodesic curvature of any curve on the surface is the ordinary curvature. It may be proved that, when the linear element of a surface is given by $ds^2 = E(du^2 + dv^2)$, the geodesic curvature, $1/\rho_\phi$, of the curve on the surface whose equation is $\phi(u, v) = 0$ is given by*

$$-\frac{1}{\rho_\phi} = \frac{1}{E}\left\{\frac{\partial}{\partial u}\frac{\sqrt{E}\,\dfrac{\partial \phi}{\partial u}}{\sqrt{\left(\dfrac{\partial\phi}{\partial u}\right)^2 + \left(\dfrac{\partial\phi}{\partial v}\right)^2}} + \frac{\partial}{\partial v}\frac{\sqrt{E}\,\dfrac{\partial \phi}{\partial v}}{\sqrt{\left(\dfrac{\partial\phi}{\partial u}\right)^2 + \left(\dfrac{\partial\phi}{\partial v}\right)^2}}\right\}.$$

Suppose a map constructed in the plane S', whose linear element is given by $ds'^2 = E'(du^2 + dv^2)$, where as in the previous discussion

$$u' \pm iv' = f(u + iv) \quad \text{and} \quad E' = f'(u + iv)\bar{f}'(u - iv).$$

Let us introduce a variable, τ, defined by the equations

$$\frac{\dfrac{\partial \phi}{\partial u}}{\sqrt{\left(\dfrac{\partial\phi}{\partial u}\right)^2 + \left(\dfrac{\partial\phi}{\partial v}\right)^2}} = \cos\tau, \qquad \frac{\dfrac{\partial \phi}{\partial v}}{\sqrt{\left(\dfrac{\partial\phi}{\partial u}\right)^2 + \left(\dfrac{\partial\phi}{\partial v}\right)^2}} = \sin\tau.$$

Then we may write for a geodesic line of the surface, whose equation is $\phi(u, v) = 0$, and for the curvature, $1/\rho$, of its representation on the map S', which has the same equation,

$$0 = \frac{1}{E}\left\{\frac{\partial}{\partial u}(\sqrt{E}\cos\tau) + \frac{\partial}{\partial v}(\sqrt{E}\sin\tau)\right\},$$

$$-\frac{1}{\rho} = \frac{1}{E'}\left\{\frac{\partial}{\partial u}(\sqrt{E'}\cos\tau) + \frac{\partial}{\partial v}(\sqrt{E'}\sin\tau)\right\}.$$

If we perform the indicated differentiations, we may eliminate the derivatives of τ, and obtain the result

$$-\frac{1}{\rho} = \frac{1}{\sqrt{E'}}\left\{\left(\frac{1}{\sqrt{E'}}\frac{\partial\sqrt{E'}}{\partial u} - \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial u}\right)\cos\tau + \left(\frac{1}{\sqrt{E'}}\frac{\partial\sqrt{E'}}{\partial v} - \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial v}\right)\sin\tau\right\}.$$

* For proof of this formula and the statements concerning geodesic curvature see Bianchi-Lukat, *Differentialgeometrie*, chap. VI.

The maximum value at a point (u, v) of $1/\rho^2$, found by varying τ, is easily seen to be

$$\frac{1}{E'}\left(\frac{1}{\sqrt{E'}}\frac{\partial\sqrt{E'}}{\partial u}-\frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial u}\right)^2+\frac{1}{E'}\left(\frac{1}{\sqrt{E'}}\frac{\partial\sqrt{E'}}{\partial v}-\frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial v}\right)^2,$$

which we will write $1/r^2$, and which may be transformed into the expression

$$\frac{1}{r^2}=\frac{E}{E'^2}\left\{\left(\frac{\partial}{\partial u}\sqrt{\frac{E'}{E}}\right)^2+\left(\frac{\partial}{\partial v}\sqrt{\frac{E'}{E}}\right)^2\right\}.$$

We have now the scale of the map given by

$$m=\frac{ds'}{ds}=\sqrt{\frac{E'}{E}},\text{ and consequently }\frac{E}{E'^2}=\frac{1}{E'm^2},$$

so that we may write

$$\frac{1}{r^2}=\frac{1}{E'm^2}\left\{\left(\frac{\partial m}{\partial u}\right)^2+\left(\frac{\partial m}{\partial v}\right)^2\right\}=\frac{1}{E'}\left\{\left(\frac{\partial \log m}{\partial u}\right)^2+\left(\frac{\partial \log m}{\partial v}\right)^2\right\}.$$

The harmonic function U is to be chosen so that the mean value of $1/r^2$ on the map shall be a minimum. We must then make a minimum the integral over the map

$$\iint\frac{du'\,dv'}{r^2}=\iint\frac{E'\,du\,dv}{r^2}=\iint\left\{\left(\frac{\partial \log m}{\partial u}\right)^2+\left(\frac{\partial \log m}{\partial v}\right)^2\right\}du\,dv.$$

Now we have $\log m = U - \frac{1}{2}\log E$, and E is a given function of u and v so that any variation of $\log m$ is a variation of U alone and consequently itself a harmonic function. Let any such variation be represented by q. Then a necessary condition that the integral be a minimum is the vanishing of the first variation of the integral so that

$$\iint\left[\frac{\partial \log m}{\partial u}\frac{\partial q}{\partial u}+\frac{\partial \log m}{\partial v}\frac{\partial q}{\partial v}\right]du\,dv=0.$$

This integral may, by Green's theorem,* be written

$$\int \log m \frac{\partial q}{\partial n} \cdot ds - \int\int \log m \cdot \Delta q \cdot du \cdot dv = 0,$$

where n denotes the exterior normal and the simple integral is taken around the boundary of the map in the uv plane, S_1. Now since q is a harmonic function, Δq vanishes, and the condition becomes

$$\int \log m \cdot \frac{\partial q}{\partial n} \cdot ds = 0.$$

We know that for any harmonic function the integral

$$\int \frac{\partial q}{\partial n} ds = 0,$$

and from this and the condition found Eisenlohr infers immediately that $\log m$ must be constant on the boundary. This conclusion does not appear to me wholly obvious, but may be reached, when the map is simply connected, as follows: Let the harmonic function conjugate to q be p. Then we know

$$\frac{\partial q}{\partial n} = \frac{\partial p}{\partial s}$$

and

$$\int \log m \frac{\partial q}{\partial n} \cdot ds = \int \log m \frac{\partial p}{\partial s} \cdot ds$$

$$= \int \left\{ \frac{\partial (p \log m)}{\partial s} - p \frac{\partial \log m}{\partial s} \right\} ds.$$

If the map on the plane S_1 is simply connected, p is single valued† and

$$\int \frac{\partial (p \log m)}{\partial s} ds = 0.$$

* See, for example, Picard, *Traité d'analyse*, vol. 2, p. 10.

† Osgood, *Functionentheorie*, p. 546.

In that case the condition becomes

$$\int p \frac{\partial \log m}{\partial s} ds = 0.$$

Again if the region is simply connected a harmonic function p may be found taking any continuous set of boundary values,* and the variation q chosen as the single valued function conjugate to p, and consequently the condition requires that at every point of the boundary

$$\frac{\partial \log m}{\partial s} = 0,$$

and m must be constant on the boundary. It is to be noticed that this condition is not proved sufficient to make the integral a minimum, and that consequently the principle of Eisenlohr is less complete than that of Tchébychef.

* By the solution of "Dirichlet's problem." See Picard, loc. cit., vol. 2, p. 88. We must require further that the boundary consist of a finite number of "regular" arcs.

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